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Phase transition for perfect condensation and instability under the perturbations on jump rates of the zero-range process

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Abstract

Let $Z \doteq (Z_1, Z_2, \ldots, Z_n)$ represent the steady state of a zero-range process in which n sites are occupied by m particles, with a jump rate between sites given by g. If $m = n$ (a particle density of 1) and Z_n^* is the maximum cluster size, perfect condensation occurs if $n - Z_n^*$ converges to 0 in probability as n tends to infinity. In this paper, we improve the description of the conditions for perfect condensation, first introduced by Jeon *et al* (2000 *Ann. Prob.* **28** 1162) and Jeon and March (2000 *Stochastic Models. Proc. Int. Conf. on Stochastic Models in Honor of Professor Donald A Dawson (Ottawa, Canada, 10–13 June 1998)* p 233). Applying the results to a few special cases, we demonstrate the existence of an interesting phase transition and conclude that the maximum cluster size in a zero-range process is unstable with respect to fluctuations in the jump rate, g.

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1. Introduction

A zero-range process is a system of n interacting particles distributed over m sites, allowing for multiple occupancy of each site. The occupancy times (length of time that a particle spends in a given site) are exponentially distributed with the parameter $g(k)$, which depends only on the total number of particles k at the site, and the particles jump to new sites according to a given probability distribution. It is customary to interpret the k particles at a site as a k-cluster. Zero-range processes were first introduced by Spitzer [3], although the growth of clusters was not described until 30 years later. In the field of mathematics, the growth of clusters in zero-range processes was studied by Jeon, March and Pittel [1] as well as Jeon and March [2]. The discussion presented in [1] considered cases in which the jump rate was given by

$$g(k) = 1 + \frac{\beta}{k}, \qquad \beta > 0. \tag{1.1}$$

This work demonstrated the existence of a condensation transition (theorem 2.2 in [1]).

Condensation transitions in zero-range processes were studied independently, in the field of physics, by Evans for the conditions given in (1.1) [4]. After the discovery of the condensation transition, several physicists applied the study of zero-range processes to sandpile dynamics, interface growth, granular systems, network flows and transport processes. These developments and applications are discussed in [5].

In [1], Jeon, March and Pittel also considered cases in which the rate function is given by

$$g(k) = k^{-\alpha}, \qquad -\infty < \alpha < \infty, \tag{1.2}$$

and showed that if the transition matrix $\{P_{ij}\}_{i,j=1}^{n}$ is symmetric and irreducible, then two striking transitions, with respect to the size of the largest cluster, are present under invariant measures.

Assume that $m = n$, i.e. the density $\rho = m/n = 1$, and let $Z \doteq (Z_1, Z_2, \cdots, Z_n)$ be a random vector corresponding to the invariant measure with a jump rate g and

$$Z_n^* = \max_{1\leqslant i\leqslant n} Z_i. \tag{1.3}$$

Then, we have the following theorem [1].

Theorem 1.1.

(a) If $\alpha > 1$, then $n - Z_n^*$ converges to 0 in probability.
(b) If $\alpha = 1$, then $n - Z_n^*$ weakly converges to a Poisson distribution with the parameter equal to 1.
(c) If $0 < \alpha < 1$, then $(n - Z_n^*)/n^{1-\alpha}$ converges to 1 in probability.
(d) If $\alpha = 0$, then $Z_n^*/\log n$ converges to $\log 2$ in probability.
(e) If $\alpha < 0$, then $Z_n^* \log(\log n)/\log n$ converges to $-\alpha^{-1}$ in probability.

In the above theorem, (a) implies that all particles coalesce to form a single cluster if $\alpha > 1$. (b) and (c) imply that the cluster loses particles if $\alpha \leqslant 1$, and an interesting transition results. In the context of physics discussions, 'condensation' is generally defined in terms of the maximum cluster size with respect to a positive fraction of the number of total particles, and 'complete condensation' means that the maximum cluster size is of order n [6]. In this discussion, we use 'perfect condensation' to mean that all particles coalesce into a single cluster. In [2], we used the term 'condensation' for 'perfect condensation'. To reduce confusion, we introduce these concepts precisely.

Definition 1.2.

(1) A condensation event occurs if Z_n^*/n converges to a positive constant, <1, in probability as n tends to infinity.
(2) A complete condensation event occurs if Z_n^*/n converges to 1 in probability as n tends to infinity.
(3) A perfect condensation event occurs if $n - Z_n^*$ converges to 0 in probability as n tends to infinity.

The properties of perfect condensation depend strongly on the tightness of the sequence, which is given by the following definition.

Definition 1.3. *The sequence* $(n - Z_n^*, n \geqslant 1)$ *is tight if, for any* $\epsilon > 0$, *N and K exist such that*

$$P\{n - Z_n^* \geqslant K\} < \epsilon, \tag{1.4}$$

for all $n > N$.

The relation between tightness and perfect condensation was described in [2].

Theorem 1.4. *Perfect condensation occurs if and only if the sequence* $(n - Z_n^*, n \geqslant 1)$ *is tight and* $ng(n) \to 0$.

Our aim here is to generalize the results in [2] and to find more relaxed conditions for tightness. These results will facilitate a detailed understanding of the process of perfect condensation.

In systems of many particles, the behavior of small numbers of particles has not traditionally been studied. Recently, advances in the study of complex systems have necessitated a higher level of detail in the description of these systems. Zero-range processes have, in particular, been applied to network problems, and complete condensation has been studied in this context [6].

Assume that $g(k)$ is given by

$$g(k) = \frac{M}{k^\alpha}, \qquad \alpha > 1; \tag{1.5}$$

then theorem 3.5 in [2] guarantees that $(n - Z_n^*, n \geqslant 1)$ is tight. Therefore, from theorem 1.4 we see that perfect condensation occurs. The determination of tightness, however, is not an easy task. By replacing M in (1.5) with a nontrivial function $M(k)$ of k, which is bounded by $M_1 \leqslant M(k) \leqslant M_2$, for some positive constants M_1, M_2, we provide an interesting example of jump rates for which the condition $ng(n) \to 0$ is satisfied, but $(n - Z_n^*, n \geqslant 1)$ is not tight. This example is important, in that it demonstrates that tightness is significantly affected by perturbation of the jump rates.

In this study of the complex system driven by such a deterministic perturbation, we show that a critical rate exists that distinguishes perfect condensation. One striking result is that perfect condensation may not be preserved under small fluctuations near a critical jump rate (theorems 4.2 and 4.3), even deep within regions of perfect condensation. This result indicates the unstable nature of clustering dynamics in zero-range processes. Note that instability of condensation in zero-range processes was studied by Grosskinsky, Chleboun and Schütz [7]. They showed that a random perturbation of the jump rates changes the critical behavior drastically.

This study is organized as follows: section 2 briefly introduces the study of zero-range processes and invariant measures, section 3 presents a proof of the main theorems describing the tightness of the sequence $(n - Z_n^*, n \geqslant 1)$ and section 4 describes the applications of these results to special cases.

2. Zero-range processes

Let $N_n = \{1, 2, \ldots, n\}$ represent a lattice with periodic boundary conditions, and let the configuration space be given by $\Omega_n^* = \{0, 1, 2, \ldots\}^{N_n}$. Assume that there is a stochastic matrix $\{P_{ij}\}_{1 \leqslant i,j \leqslant n}$, with $P_{ij} = P_{ji}$ and $\sum_{j=1}^n P_{ij} = 1$ for all i, which makes the Markov chain defined on (2.1) irreducible. Let g be a nonnegative function of nonnegative integers with $g(0) = 0$, which represents the jump rate. A zero-range process is a stochastic process defined on Ω_n^* with the following dynamics. Assume that the process is in the state η at a particular time, which suggests that at site i there is an $\eta(i)$-cluster. The $\eta(i)$-cluster is present at a given site, i, for an exponentially distributed length of time according to $g(\eta(i))$, and then allocates a particle to the cluster at a new site j with the probability P_{ij}. This reduces $\eta(i)$ to $\eta(i) - 1$, and $\eta(j)$ increases to $\eta(j) + 1$. Note that these dynamics do not permit the creation or annihilation of particles.

Let $\eta_t \doteq (\eta_t(1), \eta_t(2), \ldots, \eta_t(n))$, $0 \leqslant t < \infty$, be a Markov process that represents such dynamics. Because η_t preserves the total number of particles, i.e. $\sum_{i=1}^n \eta_t(i) = \sum_{i=1}^n \eta_0(i)$ for all t, and because P_{ij} is irreducible, if we let

$$\Omega_n^m = \left\{ \eta \in \Omega_n^* : \sum_{i=1}^n \eta(i) = m \right\}, \qquad 1 \leqslant m < \infty, \tag{2.1}$$

then there is a unique invariant measure on Ω_n^m, say ν_n^m, that produces a steady state in the process. Let $Z \doteq (Z_1, Z_2, \ldots, Z_n)$ be a random vector corresponding to the invariant measure. The following lemma defines an explicit invariant measure on Ω_n^m.

Lemma 2.1 (Spitzer [3]). *For any jump rate $g(l)$, and for any $\eta \in \Omega_n^m$, let*

$$\mu_n^m(\eta) = \prod_{i=1}^n \{g!(\eta(i))\}^{-1}, \tag{2.2}$$

where $g!(l) = g(l)g(l-1)g(l-2)\cdots g(1)$, with the convention $g!(0) = 1$. Let

$$\nu_n^m(\eta) = \frac{1}{\Gamma}\mu_n^m(\eta), \tag{2.3}$$

where Γ is the normalizing constant given by $\Gamma = \mu_n^m\left(\Omega_n^m\right) = \sum_{\eta \in \Omega_n^m} \mu_n^m(\eta)$. Then ν_n^m is the equilibrium measure corresponding to $g(l)$.

Let $\left|\Omega_n^m\right|$ be the number of elements in Ω_n^m. Then, since Ω_n^m is the set of nonnegative integers satisfying the equation

$$x_1 + x_2 + \cdots + x_n = m,$$

elementary combinatorics gives

Lemma 2.2.

$$\left|\Omega_n^m\right| = \binom{n+m-1}{n-1}.$$

3. Main theorems and proofs

For simplicity, we will consider only the case in which $m = n$, i.e. systems with a density of 1. The general case of $m/n \to \rho(>0)$ as $m, n \to \infty$ is similar. To simplify the notation, we will indicate $\Omega_n^m, \mu_n^m, \nu_n^m$ as Ω_n, μ_n, ν_n, respectively.

Since the configuration space consists of nonnegative integer partitions of n, the equilibrium measure is a random measure on the partitions. To analyze the random structure, we introduce a family of independent and identically distributed random variables $\{X_i\}_{i=1}^n$ on $\{0, 1, \ldots\}$ defined by

$$P\{X_i = k\} = \frac{x^k}{\tilde{\Gamma} g!(k)},$$

for $x \in R$. Here, $\tilde{\Gamma}$ is the normalizing constant. One can easily show that

$$(Z_1, Z_2, \ldots, Z_n) =_d (X_1, X_2, \ldots, X_n | X_1 + X_2 + \cdots + X_n = n), \tag{3.1}$$

where $=_d$ indicates that both terms are equal in distribution. That is, for any $A_1, A_2, \ldots,$ $A_n \subset R$,

$$P\{Z_1 \in A_1, Z_2 \in A_2, \ldots, Z_n \in A_n\} \tag{3.2}$$

$$= \frac{P\{X_1 \in A_1, X_2 \in A_2, \ldots, X_n \in A_n, X_1 + X_2 + \cdots + X_n = n\}}{P\{X_1 + X_2 + \cdots + X_n = n\}}. \tag{3.3}$$

The goal is to choose x to maximize the denominator in (3.3) and successfully remove the dependence structure. This type of conditioning device has been used in studies of random partitions by many investigators, see references in [1].

Recall that $Z_n^* = \max_{1 \leqslant i \leqslant n} Z_i$.

Theorem 3.1. *Let A_n be the logarithmic average of g given by*

$$A_n = \frac{1}{n} \sum_{i=1}^{n} \log(g(i)). \tag{3.4}$$

If there exist constants $C_1, C_2 > 0$ such that the following conditions hold for large n, then the sequence $(n - Z_n^*, n \geqslant 1)$ is tight:

$$\exp(A_n) \leqslant \frac{C_1}{n}, \tag{C.1}$$

$$\log(g(n)) - A_n \leqslant -C_2. \tag{C.2}$$

Proof. To prove the theorem, we construct independent and identically distributed random variables $\{X_i\}_{i=1,2,\ldots}$, with the judicious choice of

$$x = x_n = \exp\left(A_n - \frac{\log n}{n}\right), \tag{3.5}$$

as explained above. Using x_n, we define a random variable X^n by

$$P\{X^n = k\} = \frac{q_k}{\tilde{\Gamma}}, \qquad q_k = \frac{x_n^k}{g!(k)}, \qquad 0 \leqslant k \leqslant n, \tag{3.6}$$

where $\tilde{\Gamma} = \sum_{0 \leqslant k \leqslant n} q_k$ is the normalizing constant.

Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with the same distribution as X^n. Then we have

$$(Z_1, Z_2, \ldots, Z_n) =_d (X_1, X_2, \ldots, X_n | X_1 + X_2 + \cdots + X_n = n). \tag{3.7}$$

This intuitive choice of x_n makes the distribution of X^n U-shaped with $q_0 = 1, q_1 = O(1/n), q_2 = O(1/n^2), \ldots, q_n = 1/n$, which restricts the denominator in (3.3) to values greater than a constant $c > 0$.

To see this, let us investigate the asymptotic behavior of all q_k. First, since

$$q_k = \frac{x_n^k}{g!(k)}$$

$$= \frac{\exp\left(kA_n - \frac{k}{n}\log n\right)}{\exp\left(\sum_{i=1}^{k} \log(g(i))\right)}$$

$$= \frac{\exp\left(kA_n - \sum_{i=1}^{k}\log(g(i))\right)}{\exp\left(\frac{k}{n}\log n\right)}$$
$$= \frac{\exp(k(A_n - A_k))}{\exp\left(\frac{k}{n}\log n\right)},$$

we have

$$q_0 = 1,$$
$$q_1 = x_n \leqslant \exp(A_n) \leqslant \frac{C_1}{n},$$
$$q_2 = \frac{x_n^2}{g!(2)} \leqslant \frac{\exp(2A_n)}{g!(2)} \leqslant \frac{C_3}{n^2},$$

for some constant C_3. We also have

$$q_n = \frac{1}{\exp(\log n)} = \frac{1}{n}.$$

To check q_k, $3 \leqslant k \leqslant n-1$, we claim that the ratio $P_k \doteq q_{n-k}/q_n$ can be expressed as the following lemma.

Lemma 3.2.

$$P_k \doteq \frac{q_{n-k}}{q_n} = \exp\left(\frac{k}{n}\log n\right)\exp(\alpha_k), \tag{3.8}$$

where

$$\alpha_k = (n-k)\left(\frac{\log(g(n)) - A_n}{n-1} + \frac{\log(g(n-1)) - A_{n-1}}{n-2} + \cdots + \frac{\log(g(n-k+1)) - A_{n-k+1}}{n-k}\right).$$

Proof. The proof will be performed by induction on k. Let $r_k = q_k/q_{k+1}$, then

$$r_k = g(k+1)/x_n = \exp\left(\frac{1}{n}\log n\right)\exp(\log(g(k+1)) - A_n) \tag{3.9}$$

and for $k = 1$,

$$P_1 = \frac{q_{n-1}}{q_n}$$
$$= r_{n-1}$$
$$= \exp\left(\frac{1}{n}\log n\right)\exp(\log(g(n)) - A_n)$$
$$= \exp\left(\frac{1}{n}\log n\right)\exp(\alpha_1).$$

Therefore, (3.8) is true for $k = 1$. Now, assume that (3.8) is true for k. Then

$$P_{k+1} = \frac{q_{n-(k+1)}}{q_n}$$
$$= \frac{q_{n-k}}{q_n}\frac{q_{n-k-1}}{q_{n-k}}$$
$$= P_k r_{n-k-1}$$
$$= \exp\left(\frac{k+1}{n}\log n\right)\exp(\alpha_k + \log(g(n-k)) - A_n).$$

Here,

$$\alpha_k + \log(g(n-k)) - A_n = \alpha_k + (\log(g(n-k)) - A_{n-k}) + (A_{n-k} - A_{n-k+1}) + \cdots + (A_{n-1} - A_n).$$

Note that

$$\begin{aligned} A_{l-1} - A_l &= \frac{1}{l-1}\sum_{i=1}^{l-1}\log(g(i)) - \frac{1}{l}\sum_{i=1}^{l}\log(g(i)) \\ &= \frac{l\sum_{i=1}^{l-1}\log(g(i)) - (l-1)\sum_{i=1}^{l}\log(g(i))}{l(l-1)} \\ &= \frac{\sum_{i=1}^{l-1}\log(g(i)) - (l-1)\log(g(l))}{l(l-1)} \\ &= \frac{\sum_{i=1}^{l}\log(g(i)) - l\log(g(l))}{l(l-1)} \\ &= \frac{A_l - \log(g(l))}{l-1}. \end{aligned}$$

Therefore,

$$\begin{aligned} &\alpha_k + \log(g(n-k)) - A_n \\ &\quad = \alpha_k + (\log(g(n-k)) - A_{n-k}) + \frac{A_{n-k+1} - \log(g(n-k+1))}{n-k} \\ &\quad\quad + \cdots + \frac{A_n - \log(g(n))}{n-1} \\ &\quad = \frac{n-k-1}{n-1}(\log(g(n)) - A_n) + \cdots + \frac{n-k-1}{n-k-1}(\log(g(n-k)) - A_{n-k}) \\ &\quad = \alpha_{k+1}. \end{aligned}$$

That is,

$$P_{k+1} = \exp\left(\frac{k+1}{n}\log n\right)\exp(\alpha_{k+1}), \tag{3.10}$$

and the proof of lemma 3.2 is completed. □

Now, let us estimate $\tilde{\Gamma} = \sum_{i=0}^{n} q_i$ to show that it is of order $1 + O\left(\frac{1}{n}\right)$, which is needed to prove that the denominator in (3.3) is bigger than a constant $C > 0$. To do this, choose N such that conditions (C.1), (C.2) are satisfied for all $n \geqslant N$ and $C_2 N \geqslant 8$, where C_2 is the constant in (C.2). Then, for $1 \leqslant i \leqslant N$,

$$q_i = \frac{\exp(i(A_n - A_i))}{\exp\left(\frac{i}{n}\log n\right)} \tag{3.11}$$

$$\leqslant C(\exp(A_n))^i \tag{3.12}$$

$$\leqslant \frac{C}{n^i}. \tag{3.13}$$

For $N < i \leqslant n$, we subdivide the interval and apply lemma 3.2 to obtain

(i)

$$
\begin{aligned}
\sum_{N<i\leqslant\sqrt{n}} \frac{q_i}{q_n} &= \sum_{N<i\leqslant\sqrt{n}} \exp\left(\frac{n-i}{n}\log n + \alpha_{n-i}\right) \\
&\leqslant \sum_{N<i\leqslant\sqrt{n}} \exp\Bigg(\log n - i\Bigg(\frac{\log(g(n)) - A_n}{n-1} \\
&\quad + \cdots + \frac{\log(g(i+1)) - A_{i+1}}{i}\Bigg)\Bigg) \\
&\leqslant \sum_{N<i\leqslant\sqrt{n}} \exp\left(\log n - C_2 N\left(\frac{1}{n-1} + \cdots + \frac{1}{i}\right)\right) \\
&\leqslant \sqrt{n}\exp(\log n - C_2 N(\log n - \log(\sqrt{n}) + o(\log n))) \\
&\leqslant \sqrt{n}\exp(\log n - 4(\log n + o(\log n)) \\
&\leqslant \sqrt{n}\exp(-3\log n + o(\log n)) \\
&\leqslant Cn^{-2},
\end{aligned}
$$

for large n.

(ii)

$$
\begin{aligned}
\sum_{\sqrt{n}<i\leqslant n/2} \frac{q_i}{q_n} &\leqslant \sum_{\sqrt{n}<i\leqslant n/2} \exp\left(\log n - C_2\sqrt{n}\left(\frac{1}{n-1} + \cdots + \frac{1}{i}\right)\right) \\
&\leqslant \frac{n}{2}\exp\left(\log n - C_2\sqrt{n}\frac{1}{n-1}\frac{n}{2}\right) \\
&\leqslant \frac{n}{2}\exp\left(\log n - C_2\frac{\sqrt{n}}{2}\right) \\
&\leqslant Cn^{-2},
\end{aligned}
$$

for large n.

(iii)

$$
\begin{aligned}
\sum_{n/2<i\leqslant n-\sqrt{n}} \frac{q_i}{q_n} &\leqslant \sum_{n/2<i\leqslant n-\sqrt{n}} \exp\left(\log n - C_2\frac{n}{2}\left(\frac{1}{n-1} + \cdots + \frac{1}{i}\right)\right) \\
&\leqslant \frac{n}{2}\exp\left(\log n - C_2\frac{n}{2}\frac{1}{n-1}\sqrt{n}\right) \\
&\leqslant n\exp\left(-\frac{C_2}{2}\sqrt{n}\right) \\
&\leqslant Cn^{-2},
\end{aligned}
$$

for large n.

(iv)

$$
\begin{aligned}
\sum_{n-\sqrt{n}<i\leqslant n} \frac{q_i}{q_n} &\leqslant \sum_{n-\sqrt{n}<i\leqslant n} \exp\left(\frac{n-i}{n}\log n - C_2 i\left(\frac{1}{n-1} + \cdots + \frac{1}{i}\right)\right) \\
&\leqslant \sum_{n-\sqrt{n}<i\leqslant n} \exp\left(\frac{\sqrt{n}\log n}{n} - C_2 i\frac{1}{n-1}(n-i)\right)
\end{aligned}
$$

$$\leqslant \sum_{0\leqslant k<\sqrt{n}} \exp\left(-C_2 k + o(1) + O\left(\frac{\log n}{\sqrt{n}}\right)\right)$$
$$\leqslant C,$$

for some constant C independent of n, because the sum is bounded by a geometric series.

From (i)–(iv), and given that $q_n = 1/n$, we have

$$\begin{aligned}\tilde{\Gamma} &= \sum_{i=0}^{n} q_i \\ &= q_0 + (q_1 + \cdots + q_N) + \sum_{i=N+1}^{n} q_i \\ &= 1 + O\left(\frac{1}{n}\right) + O(q_n) \\ &= 1 + O\left(\frac{1}{n}\right).\end{aligned}$$

Therefore,

$$(\tilde{\Gamma})^n \leqslant \Gamma_0 < \infty$$

for some constant Γ_0. Hence

$$\begin{aligned}P\{X_1 + X_2 + \cdots + X_n = n\} &\geqslant nP\{X_1 = n, X_2 = 0, \cdots, X_n = 0\} \\ &= n\frac{q_n}{\tilde{\Gamma}}\frac{q_0^{n-1}}{(\tilde{\Gamma})^{n-1}} \\ &\geqslant \frac{1}{\Gamma_0},\end{aligned}$$

for large n, because $nq_n = 1$ and $q_0 = 1$.

Now, let $X_n^* = \max_{1\leqslant j\leqslant n} X_j$. Then, because

$$\left(Z_n^1, Z_n^2, \ldots, Z_n^n\right) =_d (X_1, X_2, \ldots, X_n | X_1 + X_2 + \ldots + X_n = n), \tag{3.14}$$

for any fixed l,

$$\begin{aligned}P\{Z_n^* \leqslant n - l\} &= \frac{P\{X_n^* \leqslant n - l, \sum X_i = n\}}{P\{X_1 + X_2 + \cdots + X_n = n\}} \\ &\leqslant \Gamma_0 P\{X_n^* \leqslant n - l\}.\end{aligned}$$

Because all X_i are independent and identically distributed,

$$P\{X_n^* = i\} \leqslant nP\{X_1 = i\}.$$

Therefore,

$$\begin{aligned}P\{X_n^* \leqslant n - l\} &= \sum_{i=1}^{n-l} P\{X_n^* = i\} \\ &= P\{X_n^* = 1\} + \sum_{2\leqslant i\leqslant n-\sqrt{n}} P\{X_n^* = i\} + \sum_{n-\sqrt{n}<i\leqslant n-l} P\{X_n^* = i\}.\end{aligned}$$

The first term $P\{X_1 = 1, \ldots, X_n = 1\} = (q_1/\tilde{\Gamma})^n$ tends to 0. It can be easily seen that the second term also tends to 0, using the results from (3.13) and (i), (ii), (iii). Indeed, for large n,

$$\sum_{2\leqslant i\leqslant n-\sqrt{n}} P\{X_n^* = i\} \leqslant n \sum_{2\leqslant i\leqslant n-\sqrt{n}} P\{X_1 = i\}$$
$$\leqslant n \sum_{2\leqslant i\leqslant n-\sqrt{n}} \frac{q_i}{\tilde{\Gamma}}$$
$$\leqslant Cn(n^{-2}) \leqslant \frac{C}{n}.$$

The last term is arbitrarily small for large l, as described in (iv), which demonstrates the tightness of $(n - Z_n^*, n \geqslant 1)$. □

If $ng(n) \to 0$ and $(n - Z_n^*, n \geqslant 1)$ is tight, then theorem 1.4 implies that perfect condensation occurs. Therefore, we have

Corollary 3.1. *Assume $ng(n) \to 0$. Then, under the conditions of theorem 3.1, perfect condensation occurs.*

4. Applications

In this section, we consider the application of the theorems presented in the previous section to the analysis of some interesting cases. Consider the case in which

$$g(n) = \frac{M}{n^\alpha}, \qquad \alpha > 0. \tag{4.1}$$

Then,

(i) From Stirling's formula,

$$A_n = \frac{1}{n}\sum_{i=1}^{n} \log\left(\frac{M}{i^\alpha}\right)$$
$$= \frac{1}{n}\log\left(\frac{M^n}{(n!)^\alpha}\right)$$
$$\sim \log M - \alpha \log n + \alpha + o(1).$$

Hence,

$$\exp(A_n) \sim \frac{C}{n^\alpha}(1 + o(1)),$$

and if $\alpha \geqslant 1$, then condition (C.1) is satisfied.

(ii) Given condition (C.2), because

$$\log(g(n)) = \log M - \alpha \log n,$$

we have

$$\log(g(n)) - A_n \sim -\alpha < 0.$$

Therefore, if $\alpha \geqslant 1$, then $g(n)$ satisfies the conditions of theorem 3.1, which implies the tightness of $(n - Z_n^*, n \geqslant 1)$. Furthermore, if $\alpha > 1$, then

$$ng(n) = \frac{M}{n^{\alpha-1}} \to 0. \tag{4.2}$$

From corollary 3.1, perfect condensation occurs, and we recover the results presented in [1].

Interesting stability behavior for zero-range processes occurs if M is replaced with a nontrivial function of n. To see this, let us consider the case in which $g(n)$ is given by

$$g(n) = \begin{cases} \dfrac{M}{n^{\alpha}}, & \text{if} \quad 2^{2k} < n \leqslant 2^{2k+1} \\ \dfrac{1}{n^{\alpha}}, & \text{if} \quad 2^{2k-1} < n \leqslant 2^{2k}, \end{cases} \tag{4.3}$$

for $k = 0, 1, \ldots$. Note that because $2g(2n) = g(n)$, g satisfies a scaling property.

Application of theorem 3.1 to this perturbed model demonstrates that if

$$\exp\left(-\frac{3}{2}\alpha\right) < M < \exp\left(\frac{3}{2}\alpha\right),$$

$(n - Z_n^*, n \geqslant 1)$ is tight. Furthermore, an interesting transition behavior to perfect condensation can be shown, and the maximum cluster size is unstable with respect to perturbations in the jump rate g, even if g is deep within the region of perfect condensation behavior. Indeed,

(i) From Stirling's formula, we have

$$\begin{aligned} A_n &\sim \frac{1}{n}\sum_{i=1}^{n} \log(g(i)) \\ &= \frac{1}{n}\left(-\sum_{i=1}^{n} \alpha \log i + [n] \log M\right), \end{aligned}$$

where $[n]$ is the number of terms of the form M/i^{α}, for $i \leqslant n$. Therefore,

$$A_n \sim -\alpha(-1 + \log n) + \frac{[n]}{n} \log M, \tag{4.4}$$

and we have

$$\exp(A_n) \leqslant \frac{C}{n^{\alpha}}.$$

(ii) Since

$$\log(g(n)) = \begin{cases} \log M - \alpha \log n, & \text{if} \quad 2^{2k} < n \leqslant 2^{2k+1} \\ -\alpha \log n, & \text{if} \quad 2^{2k-1} < n \leqslant 2^{2k}, \end{cases}$$

condition (C.2) is satisfied if

$$\log M - \alpha \log n + \alpha(-1 + \log n) - \frac{[n]}{n} \log M \leqslant -C_2$$

and

$$-\alpha \log n + \alpha(-1 + \log n) - \frac{[n]}{n} \log M \leqslant -C_2,$$

for some $C_2 > 0$. That is,

$$\left(1 - \frac{[n]}{n}\right) \log M < \alpha \qquad \text{and} \qquad \frac{[n]}{n} \log M > -\alpha.$$

Because $n/3 + o(n) \leqslant [n] \leqslant 2n/3$, we have

$$\exp\left(-\tfrac{3}{2}\alpha\right) < M < \exp\left(\tfrac{3}{2}\alpha\right).$$

From (i), (ii), we conclude that if

$$\exp\left(-\tfrac{3}{2}\alpha\right) < M < \exp\left(\tfrac{3}{2}\alpha\right), \qquad \alpha \geqslant 1,$$

then $(n - Z_n^*, n \geqslant 1)$ is tight. Corollary 3.1 implies that if $\alpha > 1$, then perfect condensation occurs. Therefore, we have

Theorem 4.1. *Assume*

$$\exp\left(-\tfrac{3}{2}\alpha\right) < M < \exp\left(\tfrac{3}{2}\alpha\right), \qquad \alpha > 1;$$

then perfect condensation occurs.

Assume that perfect condensation occurs for $M = M_1$ and $M = M_2$. We can show that perfect condensation occurs for any M between M_1 and M_2. Moreover, by the same method employed in [2], we can show that if $M \leqslant \left(\frac{1}{2}\right)^{3\alpha}$, then $(n - Z_n^*, n \geqslant 1)$ is not tight. The above theorem, together with these facts, implies the existence of a critical point around which the characteristics of the maximum cluster size of the process change.

Theorem 4.2 (Transition)**.** *For $\alpha > 1$, there is a critical point M^* satisfying*

$$\left(\tfrac{1}{2}\right)^{3\alpha} \leqslant M^* \leqslant \exp\left(-\tfrac{3}{2}\alpha\right) \tag{4.5}$$

such that perfect condensation occurs for $M > M^*$, but not for $M < M^*$.

Proof. Let $\mu_{n,M}$ be the un-normalized zero-range invariant measure corresponding to g in (4.3). Then

$$\mu_{n,M}(\eta) = \prod_{i=1}^{n} \{g!(\eta(i))\}^{-1} = \prod_{i=1}^{n} \frac{(\eta(i)!)^{\alpha}}{M^k} \tag{4.6}$$

for some $k \in \{0, 1, 2, \ldots\}$. In this expression, since k depends on η, we can define a function $\phi : \Omega_n \to \{0, 1, 2, \ldots\}$ by

$$\phi(\eta) = k. \tag{4.7}$$

Let $\eta_1 = (n, 0, \ldots, 0)$, $\eta_2 = (0, n, 0, \ldots, 0), \ldots, \eta_n = (0, \ldots, 0, n)$, and let

$$\tilde{\Omega}_n = \{\eta_1, \eta_2, \ldots, \eta_n\}$$

be the set of all configurations with an n-cluster. We also define the sets

$$\Omega_1 = \{\eta \in \Omega_n : \phi(\eta) \geqslant \phi(\eta_1)\},$$
$$\Omega_2 = \{\eta \in \Omega_n : \phi(\eta) < \phi(\eta_1)\}.$$

Note that $\Omega_1 \cup \Omega_2 = \Omega_n$ and $\Omega_1 \cap \Omega_2 = \phi$.

Assume that perfect condensation occurs for $M = M_1$ and $M = M_2$, where $M_1 < M_2$. Then, for $i = 1, 2$,

$$\frac{P\{Z_1 + \cdots + Z_n < n\}}{P\{Z_1 + \cdots + Z_n = n\}} = \frac{\mu_{n,M_i}(\Omega_n \backslash \tilde{\Omega}_n)}{\mu_{n,M_i}(\tilde{\Omega}_n)} \to 0. \tag{4.8}$$

For M_0 satisfying $M_1 < M_0 < M_2$,

$$\begin{aligned}
\frac{P\{Z_1 + \cdots + Z_n < n\}}{P\{Z_1 + \cdots + Z_n = n\}} &= \frac{\mu_{n,M_0}(\Omega_n \backslash \tilde{\Omega}_n)}{\mu_{n,M_0}(\tilde{\Omega}_n)} \\
&= \frac{\mu_{n,M_0}(\Omega_1 \backslash \tilde{\Omega}_n)}{\mu_{n,M_0}(\tilde{\Omega}_n)} + \frac{\mu_{n,M_0}(\Omega_2)}{\mu_{n,M_0}(\tilde{\Omega}_n)} \\
&\leqslant \frac{\mu_{n,M_2}(\Omega_1 \backslash \tilde{\Omega}_n)}{\mu_{n,M_2}(\tilde{\Omega}_n)} + \frac{\mu_{n,M_1}(\Omega_2)}{\mu_{n,M_1}(\tilde{\Omega}_n)}.
\end{aligned}$$

The last inequality is true from the definitions of Ω_1 and Ω_2. From (4.8), the final term tends to zero as n tends to infinity.

In [2], for the case $\alpha = 1$, we have shown that if $M \leqslant (1/2)^3$, then $(n - Z_n^*, n \geqslant 1)$ is not tight. We can easily generalize this result to the case in which $\alpha \geqslant 1$, and we conclude that if $M \leqslant (1/2)^{3\alpha}$, then $(n - Z_n^*, n \geqslant 1)$ is not tight. The existence of such a critical point is, therefore, clear. □

Note that if $\alpha > 1$, (4.1) exhibits perfect condensation. Theorem 4.2 indicates that even for large $\alpha(\gg 1)$, perturbations in M produce a phase that is not perfect condensation. This is, to some degree, counterintuitive. The natural intuition is that if g lies deep within a region of perfect condensation, then perfect condensation should be preserved under small perturbations. Our result, however, contradicts this intuition.

Moreover, we can demonstrate the existence of subsequences n_k and M in (4.3) which dictate that the fraction of the maximum cluster size $Z_{n_k}^*/n_k$ is bounded by a constant smaller than 1. In other words, the perturbed subsequence is not even within the regime of complete condensation. Indeed, let $n_k = 2^{2k}$. Then we have the following theorem.

Theorem 4.3. *For any $\epsilon > 0$, there exists M such that*

$$Z_{n_k}^*/n_k \leqslant \tfrac{1}{2} + \epsilon \tag{4.9}$$

for large n_k.

Proof. For any l, $1 \leqslant l \leqslant n_k$, let A_l be the set of configurations with the maximum cluster size greater than or equal to l. That is,

$$A_l = \{\eta \in \Omega_{n_k} : \text{ there exists } i \text{ such that } \eta(i) \geqslant l\}.$$

Note that the set $A_l \backslash A_{l+1}$ consists of all configurations in which the maximum cluster size is exactly l. Let $\eta_1 = (n_k/2, n_k/2, 0, \ldots, 0)$ and let $\eta_2 = (l, \eta(2), \eta(3), \ldots, \eta(n))$ be an element which has the maximum weight in $A_l \backslash A_{l+1}$, i.e. $\mu_{n_k}(\eta_2) \geqslant \mu_{n_k}(\eta)$ for any $\eta \in A_l \backslash A_{l+1}$. Then, for $l \geqslant (1/2 + \epsilon) n_k$,

$$\begin{aligned} P\{Z_{n_k} \in A_l \backslash A_{l+1}\} &= \nu_{n_k}(A_l \backslash A_{l+1}) \\ &\leqslant \frac{\mu_{n_k}(A_l \backslash A_{l+1})}{\mu_{n_k}(\eta_1)} \\ &\leqslant \frac{n_k \binom{2n_k - l - 1}{n_k - 1} \mu_{n_k}(\eta_2)}{\mu_{n_k}(\eta_1)}. \end{aligned}$$

The last inequality is obtained by applying lemma 2.2 to the fact that the maximum cluster size in the set $A_l \backslash A_{l+1}$ is $l > n_k/2$, the maximum cluster may be located at n_k different sites, the number of remaining particles is $n_k - l$ and $\mu_{n_k}(\eta_2) \geqslant \mu_{n_k}(\eta)$ for any $\eta \in A_l \backslash A_{l+1}$.

Because

$$\binom{2n_k - l - 1}{n_k - 1}$$

is decreasing in l, substituting l with $n_k/2 - 1$ yields

$$\binom{2n_k - l - 1}{n_k - 1} \leqslant \binom{3n_k/2}{n_k - 1}.$$

Introducing a polynomial $P(n_k)$, with a degree that is independent of n_k and l, and which may differ in each expression, we have, for $M < 1$,

$$\begin{aligned} P\{Z_{n_k} \in A_l \backslash A_{l+1}\} &\leqslant P(n_k) \binom{3n_k/2}{n_k - 1} \frac{g!(n_k/2) g!(n_k/2)}{g!(l) g!(\eta(2)) \cdots g!(\eta(n))} \\ &\leqslant P(n_k) \binom{3n_k/2}{n_k - 1} \left(\frac{l! \eta(2)! \cdots \eta(n)!}{(n_k/2)!(n_k/2)!} \right)^{\alpha} M^{l_0}, \end{aligned}$$

where $l_0 = \phi(\eta_1) - \phi(\eta_2)$, for ϕ defined in (4.7). To obtain a bound for l_0, note that since $g(n) = 1/n^{\alpha}$, if $n_k/2 < n \leqslant n_k$ and $g(n) = M/n^{\alpha}$, if $n_k/4 < n \leqslant n_k/2, \ldots$, the number of M's from l-cluster and $n_k/2$-cluster are the same as $n_k(2/3 + o(1))/2$ and the number of M's from the clusters consisting of $n_k - l$ particles is bounded by $2(n_k - l)/3$. That is,

$$\begin{aligned} \phi(\eta_1) - \phi(\eta_2) &\geqslant 2 \cdot \frac{1}{2} n_k \left(\frac{2}{3} + o(1) \right) - \frac{1}{2} n_k \left(\frac{2}{3} + o(1) \right) - \frac{2}{3}(n_k - l) \\ &\geqslant \left(\frac{1}{3} + o(1) \right) n_k - \frac{2}{3} n_k + \frac{2}{3} \left(\frac{1}{2} + \epsilon \right) n_k \\ &= \frac{2}{3} \epsilon (1 + o(1)) n_k. \end{aligned}$$

Since $\eta(2) + \cdots + \eta(n) = n_k - l$, we have $\eta(2)! \cdots \eta(n)! \leqslant (n_k - l)!$. Therefore,

$$\begin{aligned} P\{Z_{n_k} \in A_l \backslash A_{l+1}\} &\leqslant P(n_k) \binom{3n_k/2}{n_k - 1} \left(\frac{l!(n_k - l)!}{(n_k/2)!(n_k/2)!} \right)^{\alpha} M^{l_0} \\ &= P(n_k) \binom{3n_k/2}{n_k - 1} \binom{n_k}{n_k/2}^{\alpha} \binom{n_k}{l}^{-\alpha} M^{l_0} \\ &\leqslant P(n_k) \binom{3n_k/2}{n_k - 1} \binom{n_k}{n_k/2}^{\alpha} M^{l_0}. \end{aligned}$$

Applying the Stirling formula, we easily see that

$$\binom{3n_k/2}{n_k - 1} \sim P(n_k) \lambda^{n_k}$$

$$\binom{n_k}{n_k/2} \sim P(n_k) 2^{n_k},$$

where $\lambda = \sqrt{3}^3/\sqrt{2}$. Therefore,

$$\begin{aligned} P\{Z_{n_k} \in A_l \backslash A_{l+1}\} &\leqslant P(n_k) (2^{\alpha} \lambda)^{n_k} M^{2(\epsilon + o(1)) n_k/3} \\ &\leqslant P(n_k) (2^{\alpha} \lambda M^{2(\epsilon + o(1))/3})^{n_k}, \end{aligned}$$

for large n_k. The probability that the maximum cluster size is greater than or equal to $(1/2+\epsilon)n_k$ can be estimated by

$$\begin{aligned} P\left\{ Z_{n_k} \in \bigcup_{l \geqslant (1/2+\epsilon)n_k} A_l \backslash A_{l+1} \right\} &\leqslant \sum_{l \geqslant (1/2+\epsilon)n_k} P(n_k) (2^{\alpha} \lambda M^{2(\epsilon + o(1))/3})^{n_k} \\ &= P(n_k) (2^{\alpha} \lambda M^{2(\epsilon + o(1))/3})^{n_k}. \end{aligned}$$

The equality holds because there are at most $n_k/2$ terms in the summation, and the degree of $P(n_k)$, which may differ in each expression, is independent of n_k and l.

The final term decays to zero exponentially as n tends to infinity, if $M^{2\epsilon/3} < 1/(2^{\alpha}\lambda)$. As a result, the probability that the maximum cluster size is larger than $(1/2 + \epsilon)n_k$ tends to zero as n_k tends to infinity. □

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References

[1] Jeon I, March P and Pittel B 2000 *Ann. Prob.* **28** 1162
[2] Jeon I and March P 2000 *Stochastic Models. Proc. Int. Conf. on Stochastic Models in Honor of Professor Donald A Dawson* (*Ottawa, Canada,* 10–13 June 1998) p 233
[3] Spitzer F 1970 *Adv. Math.* **5** 246
[4] Evans M R 2000 *Braz. J. Phys.* **30** 42
[5] Evans M R and Hanney T 2005 *J. Phys. A: Math. Gen.* **38** R195
[6] Noh J, Shim G and Lee H 2005 *Phys. Rev. Lett.* **94** 198701
[7] Grosskinsky S, Chleboun P and Schütz G M 2008 *Phys. Rev.* E **78** 030101